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# Boundary Value Problems for the Steady Boltzmann Equation

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We discuss steady boundary value problems for the Boltzmann equation with inflow and diffusive boundary conditions in one, two, and three dimensions, with suitable truncations of the collision kernel. General existence and uniqueness results are obtained if the domain is sufficiently small. In one dimension, the existence of solutions on general intervals is obtained by abstract fixed-point theory.

**KEY WORDS:** Steady Boltzmann equation; boundary value problem; numerical simulation.

## 1. INTRODUCTION

We consider the steady Boltzmann equation

$$v \cdot \nabla_x f = Q(f), \qquad x \in \Omega, \quad v \in \mathbb{R}^3 \tag{1.1}$$

on the multidimensional domain $\Omega \subset \mathbb{R}^n$ with various boundary conditions on $\partial\Omega$ and $n = 1$, 2 or 3. Here $Q(f)$ denotes the collision operator written in the form

$$Q(f) = Q_+(f) - fL(f)$$

where $Q_+(f)$ stands for the gain term, $fL(f)$ for the loss term due to binary collision of gas particles,

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$$Q_+(f) = \int_{\mathbb{R}^3} \int_{S_+} B(|v - v_*|, n)\, f(v')\, f(v'_*)\, dn\, dv_*$$

$$L(f) = \int_{\mathbb{R}^3} \int_{S_+} B(|v - v_*|, n)\, f(v_*)\, dn\, dv_*$$

and $S_+$ is the hemisphere corresponding to $(v - v_*, n) > 0$. The pair $(v', v'_*)$ is given by the collision transformation

$$J: (v, n, v_*) \to (v', -n, v'_*)$$

by

$$v' = v - n(v - v_*, n)$$

$$v'_* = v_* + n(v - v_*, n)$$

Unless mentioned otherwise, we assume the following kind of interaction law for binary collisions: For $v \in \mathbb{R}^3$ and $n \in S^2_+$, we assume that

$$B(|v - v_*|, n) = |v - v_*|^k\, h(\theta) \tag{1.2}$$

where $\theta$ is the polar angle of $n$ relative to a polar axis in direction $v - v_*$. The function $h$ is assumed to be integrable on $[0, \pi]$ with $\int_{S_+} h(\theta)\, dn = 1$ and the integer $k$ is chosen out of the set $\{-1, 0, 1\}$, which describes Maxwellian molecules ($k = 0$), a hard-sphere gas ($k = 1$), and a soft-sphere gas ($k = -1$). Hence we cover three classical examples of molecular interaction laws.

In Section 2, we use the Kaniel–Shinbrot iteration scheme[1] to prove a general existence and uniqueness result for a truncated problem; the collision kernel is modified such that collisions in which one of the (pre- or postcollisional) velocities has modulus less than $\varepsilon$ are disregarded, and the size of the domain must be bounded in terms of $\varepsilon$.

Section 2.2 deals with precribed influx boundary conditions, whereas partial results on diffusive boundary conditions are presented in Section 2.3. In this subsection, the albedo operator is introduced and used to reduce the boundary value problem to a suitable fixed-point operator.

In Section 3, we generalize the existing global existence results for one-dimensional slabs from refs. 2 and 3 to purely diffusive boundary conditions.

Finally, in Section 4, we give some *a priori* estimates for the full, two- or three-dimensional boundary value problem for the steady Boltzmann equation, in the hope that these estimates will eventually turn out to be useful for existence theorems.

## 2. LOCAL EXISTENCE RESULTS

### 2.1. Truncation of the Collision Kernel

Due to the singular limit $v = 0$ of Eq. (1.1), we have to eliminate collisions between particles with small velocities. A formal solution of (1.1) can be derived by integrating along a characteristic line starting at some boundary point $x \in \partial\Omega$. The reason to eliminate small velocities from the collision process is to get an upper bound on the "traveling time" of a particle through the domain $\Omega$.

We introduce the following truncation of the collision kernel $B$: for $\varepsilon > 0$ arbitrary but fixed let

$$B_\varepsilon(v, v_*, n) = \begin{cases} c_\varepsilon\, |v - v_*|^k\, h(\theta) & \text{if } \min\{|v|, |v_*|, |v'|, |v'_*|\} > \varepsilon \\ 0 & \text{otherwise} \end{cases}$$

with $c_\varepsilon > 0$ such that $\int_{S_+} B_\varepsilon(v, v_*, n)\, dn = |v - v_*|$ and $k \in \{-1, 0, 1\}$. Further, let $Q^\varepsilon$ be the collision operator with $B$ replaced by $B^\varepsilon$.

The local proof is based on some *a priori* estimates on $L(f)$, where $f$ is a Maxwellian distribution. For the collision kernels $B$ and $B^\varepsilon$ given above we find the following result.

**Proposition 2.1.** Assume that $f$ is a (normalized) Maxwellian

$$f(v) = \left(\frac{\beta}{\pi}\right)^{3/2} e^{-\beta v^2}, \qquad v \in \mathbb{R}^3$$

Then

$$L(f)(v) = \begin{cases} 1 & \text{if } k = 0 \\ \dfrac{\operatorname{erf}(\sqrt{\beta}\, |v|)}{|v|} & \text{if } k = -1 \\ \dfrac{1}{\sqrt{\beta\pi}} e^{-\beta v^2} + \left(\dfrac{1}{2\beta} + |v|^2\right) \dfrac{\operatorname{erf}(\sqrt{\beta}\, |v|)}{|v|} & \text{if } k = 1 \end{cases}$$

*Proof.* Because $f$ is a (normalized) Maxwellian, we obtain, using spherical coordinates,

$$L(f)(v) = 2\pi \left(\frac{\beta}{\pi}\right)^{3/2} \int_0^\infty \int_{-1}^1 (r^2 + |v|^2 - 2r\, |v|\, x)^{k/2}\, r^2 e^{-\beta r^2}\, dx\, dr \quad (2.1)$$

and, if $k=0$, $L(f)(v)=1$ for all $v \in \mathbb{R}^3$. If $k=-1$, Eq. (2.1) reads

$$L(f)(v)=4\pi\left(\frac{\beta}{\pi}\right)^{3/2}\left[\int_0^{|v|}\frac{r^2}{|v|}e^{-\beta r^2}\,dr+\int_{|v|}^{\infty} re^{-\beta r^2}\,dr\right]$$
$$=\frac{\operatorname{erf}(\sqrt{\beta}\,|v|)}{|v|}$$

If $k=1$, Eq. (2.1) reads

$$L(f)(v)=\frac{4\pi}{3}\left(\frac{\beta}{\pi}\right)^{3/2}\left[\int_0^{|v|}\frac{r^2+3\,|v|^2}{|v|}\,r^2e^{-\beta r^2}\,dr+\int_{|v|}^{\infty}(3r^2+|v|^2)\,re^{-\beta r^2}\,dr\right]$$
$$=\frac{1}{\sqrt{\beta\pi}}e^{-\beta v^2}+\left(\frac{1}{2\beta}+|v|^2\right)\frac{\operatorname{erf}(\sqrt{\beta}\,|v|)}{|v|} \quad \blacksquare$$

**Remark 2.2.** In particular, if $v=0$,

$$L(f)(0)=\begin{cases}1 & \text{if} \quad k=0\\ 2(\beta/\pi)^{1/2} & \text{if} \quad k=-1\\ 2/\sqrt{\beta\pi} & \text{if} \quad k=1\end{cases}$$

For the truncated collision kernel $B^{\varepsilon}$ we get the following result.

**Proposition 2.3.** Assume that $f$ is a (normalized) Maxwellian

$$f(v)=\left(\frac{\beta}{\pi}\right)^{3/2}e^{-\beta v^2}, \qquad v \in \mathbb{R}^3$$

and $\varepsilon>0$; then

$$L^{\varepsilon}(f)(v)=L(f)(v)+\Delta_f$$

where

$$\Delta_f=\begin{cases}2\varepsilon\left(\frac{\beta}{\pi}\right)^{1/2}e^{-\beta\varepsilon^2}-\operatorname{erf}(\sqrt{\beta}\,\varepsilon) & \text{if} \quad k=0\\ \frac{1}{|v|}\left[2\varepsilon\left(\frac{\beta}{\pi}\right)^{1/2}e^{-\beta\varepsilon^2}-\operatorname{erf}(\sqrt{\beta}\,\varepsilon)\right] & \text{if} \quad k=-1\\ \frac{1}{|v|}\left[\left(\frac{\beta}{\pi}\right)^{1/2}\varepsilon\left(2\,|v|^2+\frac{2}{3}\varepsilon^2+\frac{1}{\beta}\right)e^{-\beta\varepsilon^2}-\left(\frac{1}{2\beta}+|v|^2\right)\operatorname{erf}(\sqrt{\beta}\,\varepsilon)\right] & \\ \quad \text{if} \quad k=1 & \end{cases}$$

*Proof.* Using the same technique as given above, the results are obtained by changing the limits of the outer integral on the right-hand side (2.1) to the interval $[\varepsilon, \infty]$. ∎

**Remark 2.4.** The explicit formula given in Proposition 2.3 is used in the following subsection to show that the "beginning conditions" of the Kaniel–Shinbrot iteration scheme cat be satisfied if the domain $\Omega$ is sufficiently small.

## 2.2. Existence Theorem for Prescribed Boundary Values

Consider for given $\varepsilon > 0$ the boundary value problem

$$v \cdot \nabla_x f + f L^\varepsilon(f) = Q^\varepsilon_+(f) \tag{2.2}$$

with some instream conditions at the boundary $\partial\Omega$

$$f(x, v) = \Phi(x, v), \qquad x \in \partial\Omega, \quad v \cdot n(x) > 0 \tag{2.3}$$

We assume that we have upper and lower bounds on $\Phi$ of the type

$$C_1 e^{-\gamma v^2} \leqslant \Phi(x, v) \leqslant C_2 e^{-\alpha v^2} \tag{2.4}$$

For simplicity we use in the following the notation $L = L^\varepsilon$ and $Q_+ = Q^\varepsilon_+$.

**Theorem 2.5.** For any $\varepsilon > 0$, $k = -1, 0$, or 1, and $\Omega(\varepsilon)$ sufficiently small, there exists a solution of the boundary value problem (2.2)–(2.4). The smallness condition on $\Omega$ is of the type

$$\operatorname{diam}(\Omega) \leqslant C \cdot \varepsilon$$

where the constant depends on $\Phi$.

*Proof.* We construct a solution via the Kaniel–Shinbrot iteration scheme.[1] Define two sequences $\{l_n\}_{n \in \mathbb{N}}$ and $\{u_n\}_{n \in \mathbb{N}}$ by

$$v \cdot \nabla_x l_{n+1} + l_{n+1} L(u_n) = Q_+(l_n) \tag{2.5}$$

$$v \cdot \nabla_x u_{n+1} + u_{n+1} L(l_n) = Q_+(u_n) \tag{2.6}$$

together with boundary conditions

$$l_{n+1} = u_{n+1} = \Phi, \qquad x \in \partial\Omega, \quad v \cdot n(x) > 0$$

and let $l_0 = l_0(v)$ and $u_0 = u_0(v)$ be two global Maxwellians with $l_0(v) < \Phi(x, v) < u_0(v)$ for all $x \in \partial\Omega$. A straightforward induction shows that the iteration defined by (2.5)–(2.6) leads to monotone and bounded sequences in the form

$$0 \leqslant l_0 \leqslant l_1 \leqslant l_2 \leqslant \cdots \leqslant l_n \leqslant u_n \leqslant u_{n-1} \leqslant \cdots \leqslant u_0$$

as long as we are able to choose $l_0$ and $u_0$ such that the "beginning conditions"

$$l_0 \leqslant l_1 \tag{2.7}$$

and

$$u_1 \leqslant u_0 \tag{2.8}$$

will be satisfied. If $|v| < \varepsilon$, we simply get

$$l_n(x, v) = \Phi(z(x, v), v) = u_n(x, v) \qquad \forall n \in \mathbb{N}$$

where

$$z(x, v) = x - tv \in \partial\Omega, \qquad t = \inf_{\tau > 0} \{\tau; x - \tau v \in \partial\Omega\}$$

What remains is to investigate the case where $|v| > \varepsilon$.

The equations for $l_1$ and $u_1$ read

$$v \cdot \nabla_x l_1 + l_1 L(u_0) = Q_+(l_0) \tag{2.9}$$

$$v \cdot \nabla_x u_1 + u_1 L(l_0) = Q_+(u_0) \tag{2.10}$$

Because we assume that $l_0$ and $u_0$ are (global) Maxwellians, we have

$$Q_+(l_0) = l_0 L(l_0)$$

$$Q_+(u_0) = u_0 L(u_0)$$

Denote $L_0 = L(l_0)$ and $U_0 = L(u_0)$. Integrating Eqs. (2.9) and (2.10) along a characteristic yields

$$l_1(x, v) = e^{-U_0 s(x, v)} \Phi(x - s(x, v)\, v, v) + (1 - e^{-U_0 s(x, v)}) \frac{L_0 l_0}{U_0}$$

$$= \frac{L_0}{U_0} l_0 + e^{-U_0 s(x, v)} \left( \Phi(x - s(x, v)\, v, v) - \frac{L_0}{U_0} l_0 \right)$$

and

$$u_1(x, v) = e^{-L_0 s(x, v)} \Phi(x - s(x, v)\, v, v) + (1 - e^{-L_0 s(x, v)}) \frac{U_0 u_0}{L_0}$$

$$= \frac{U_0}{L_0} u_0 + e^{-L_0 s(x, v)} \left( \Phi(x - s(x, v)\, v, v) - \frac{U_0}{L_0} u_0 \right)$$

with $L_0 = L_0(v)$ and $U_0 = U_0(v)$.

Here $s(x, v)$ denotes the "time" a particle with velocity $v$ needs to move from a boundary point $x - s(x, v)\, v$ to $x$,[4] i.e.,

$$s(x, v) = \inf_{\tau > 0} \{ \tau; x - \tau v \in \partial\Omega, v \cdot n(x - \tau v) > 0 \}$$

Inequalities (2.7) and (2.8) will be satisfied if

$$e^{-U_0 s(x, v)} \geqslant \frac{U_0 - L_0}{(\Phi / l_0)\, U_0 - L_0} \tag{2.11}$$

and

$$e^{-L_0 s(x, v)} \geqslant \frac{U_0 - L_0}{U_0 - (\Phi / u_0)\, L_0} \tag{2.12}$$

which should hold for all $x \in \bar{\Omega}$ and $v \in \mathbb{R}^3$ with $|v| > \varepsilon$.

The right-hand side of (2.11) can be estimated by

$$\frac{U_0 - L_0}{(\Phi / l_0)\, U_0 - L_0} \leqslant \frac{l_0}{\Phi}$$

Hence, assuming that $l_0$ and $u_0$ are given by

$$l_0 = c_1 e^{-\gamma v^2} \tag{2.13}$$

$$u_0 = c_2 e^{-\alpha v^2} \tag{2.14}$$

with $c_1 < C_1$ and $c_2 > C_2$ according to (2.4), we can choose, for a given $c_2$, $c_1$ sufficiently small to fulfill (2.11) as long as $s(x, v)\, U_0$ is uniformly bounded with respect to $x \in \Omega$ and $v \in \mathbb{R}^3$, $|v| > \varepsilon$.

Condition (2.12) turns out to be more restrictive: using the estimates

$$e^{-L_0 s(x, v)} \geqslant 1 - L_0 s(x, v)$$

and

$$\frac{U_0 - L_0}{U_0 - (\Phi/u_0)\, L_0} \leqslant 1 + \left(\frac{\Phi}{u_0} - 1\right) \frac{L_0}{U_0}$$

we see that (2.12) will hold, while

$$s(x, v) \leqslant \left(1 - \frac{\Phi}{u_0}\right) \frac{1}{U_0}$$

which obviously restricts the size of $\Omega$. Because we use the truncated collision kernel as introduced in Section 2.1, we have

$$s(x, v) \leqslant \frac{\operatorname{diam}(\Omega)}{\varepsilon} \qquad \forall x \in \Omega, \quad v \in \mathbb{R}^3, \quad |v| > \varepsilon$$

where $\operatorname{diam}(\Omega)$ is the maximal distance between two boundary points. So we are able to satisfy the "beginning conditions" for $l_0$ and $u_0$ if

$$\operatorname{diam}(\Omega) \leqslant \left(1 - \frac{\Phi}{u_0}\right) \frac{\varepsilon}{U_0} \tag{2.15}$$

where we still have to fix the constant $c_2$.

For the following estimates we use the explicit formulas for $L_0$ and $U_0$ as given by Proposition 2.3 to show that $s(x, v)\, U_0$ is uniformly bounded and (2.15) can be satisfied for a sufficiently small domain $\Omega$.

In the simplest case, i.e., $k = 0$ (Maxwellian molecules), $L_0$ and $U_0$ are uniformly bounded for $x \in \Omega$ and $v \in \mathbb{R}^3$, $|v| > \varepsilon$, by Proposition 2.3.

Hence $s(x, v)\, U_0$ is uniformly bounded and the restriction on the domain $\Omega$ reads

$$\operatorname{diam}(\Omega) \leqslant \left(\frac{\alpha}{\pi}\right)^{3/2} \frac{\varepsilon}{4C_2}$$

with $c_2 = 2C_2$.

If $k = -1$, we estimate, using Proposition (2.3),

$$U_0 \leqslant \frac{4}{\alpha} \pi c_2$$

and therefore $s(x, v)\, U_0$ is uniformly bounded for $x \in \Omega$ and $v \in \mathbb{R}^3$, $|v| > \varepsilon$. The restriction on $\Omega$ reads

$$\operatorname{diam}(\Omega) \leqslant \frac{\alpha}{\pi} \frac{\varepsilon}{8C_2}$$

with $c_2 = 2C_2$.

Finally, if $k=1$, we estimate

$$s(x, v)\, U_0 \leqslant c_2 \frac{\operatorname{diam}(\Omega)}{\varepsilon}(M_1+\varepsilon M_2+\varepsilon^2 M_3)$$

where $M_1, M_2$, and $M_3$ are some constants depending only on $\alpha$, respectively $\gamma$. Furthermore, we have

$$s(x, v) \leqslant \frac{\operatorname{diam}(\Omega)}{|v|} \leqslant \left(1-\frac{\Phi}{u_0}\right)\frac{1}{U_0}$$

which should hold for all $v \in \mathbb{R}^3$, $|v|>\varepsilon$. Because $|v|/U_0$ is strictly positive and monotonically increasing with $|v|$, we get

$$\operatorname{diam}(\Omega) \leqslant \left(\frac{\alpha}{\pi}\right)^{3/2} \frac{\varepsilon}{2U_0}$$

with $c_2 = 2C_2$.

The two sequences $\{l_n\}_{n\in\mathbb{N}}$ and $\{u_n\}_{n\in\mathbb{N}}$ are monotone and bounded and therefore convergent. If we assume that

$$l = \lim_{n\to\infty} l_n$$

and

$$u = \lim_{n\to\infty} u_n$$

we find that

$$v\cdot\nabla_x l + lL(u) = Q_+(l)$$

$$v\cdot\nabla_x u + uL(l) = Q_+(u)$$

together with the boundary condition

$$u - l = 0, \qquad x\in\partial\Omega, \quad v\cdot n(x)>0$$

It remains to show that $l=u$.

Define $h(x, v) = u(x, v) - l(x, v)$. Then the equation for $h$ reads

$$v\cdot\nabla h + hL(l) - lL(h) = Q_+(u) - Q_+(l)$$

Integrating over $\mathbb{R}^3$ and the domain $\Omega$ gives

$$\int_\Omega \int_{\mathbb{R}^3} v\cdot\nabla h\, dv\, dx = \int_\Omega \int_{\mathbb{R}^3} [Q_+(u) - Q_+(l) + lL(h) - hL(l)]\, dv\, dx$$

Because

$$\int_{\Omega}\int_{\mathbb{R}^3} [Q_+(u) - Q_+(l)]\, dv\, dx = \int_{\Omega}\int_{\mathbb{R}^3} [uL(u) - lL(l)]\, dv\, dx$$

we get

$$\int_{\Omega}\int_{\mathbb{R}^3} v \cdot \nabla h\, dv\, dx = \int_{\Omega}\int_{\mathbb{R}^3} (u + l)\, L(h)\, dv\, dx$$

Applying the divergence theorem yields

$$-\int_{\partial\Omega}\int_{v \cdot n < 0} |v \cdot n|\, h(x, v)\, dv\, d\sigma(x) = \int_{\Omega}\int_{\mathbb{R}^3} (u + l)\, L(h)\, dv\, dx$$

Because $h(x, v) \geqslant 0$ and therefore $L(h)(x, v) \geqslant 0$, we get that $h(x, v) = 0$ a.e. on $\Omega \times \mathbb{R}^3$, which completes the proof.

**Remark 2.6.** The restrictions on the size of the domain $\Omega$ depend —by construction—strongly on the truncation parameter $\varepsilon$. There is also a strong dependence on the given boundary flux $\Phi$. If $\Phi$ is sufficiently small, i.e., if the gas conditions are near to a vacuum, the restrictions on $\Omega$ become weaker.

Within the given upper and lower bounds the solution of Theorem 2.5 is unique.

**Theorem 2.7.** Suppose $\varepsilon > 0$ given and $g$ is a solution of (2.2)–(2.4) with

$$l_0(v) \leqslant g(x, v) \leqslant u_0(v) \qquad \forall x \in \Omega, \quad v \in \mathbb{R}^3$$

where $l_0$ and $u_0$ are the lower and upper bounds, respectively on the solution $f$ given by Theorem 2.5. Then $f = g$ almost everywhere.

*Proof.* Because $g$ is a solution, we have $v \cdot \nabla_x g = Q_+(g) - gL(g)$. Suppose that $l_n$ and $u_n$ are the $n$th iterations as formulated in Theorem 2.5. Then

$$v \cdot \nabla_x(g - l_n) + (g - l_n)\, L(u_{n-1}) = gL(u_{n-1} - g) + Q_+(g) - Q_+(l_{n-1})$$

$$v \cdot \nabla_x(u_n - g) + (u_n - g)\, L(l_{n-1}) = gL(g - l_{n-1}) + Q_+(u_{n-1}) - Q_+(g)$$

and with $l_0 \leqslant g \leqslant u_0$ we get inductively

$$l_n \leqslant g \leqslant u_n \qquad \forall n \in \mathbb{N}$$

By Theorem 2.5 both sequences $\{l_n\}_{n\in\mathbb{N}}$ and $\{u_n\}_{n\in\mathbb{N}}$ are convergent with

$$\lim_{n\to\infty} l_n = f = \lim_{n\to\infty} u_n$$

hence $f = g$ almost everywhere. ▮

## 2.3. Existence Theorems for Diffusive Boundary Conditions

For diffusive boundary conditions we consider for any $\varepsilon > 0$ the steady Boltzmann equation (2.2) together with the boundary condition

$$f(x, v) = m_{\beta(x)}(v) \int_{v_* \cdot n < 0} |v_* \cdot n|\, f(x, v_*)\, dv_* \qquad \forall x \in \partial\Omega, \quad v \cdot n > 0 \tag{2.16}$$

where

$$m_{\beta(x)}(v) = \frac{2\beta^2(x)}{\pi}\, e^{-\beta(x) v^2} \tag{2.17}$$

and $\alpha \leqslant \beta(x) \leqslant \gamma$, $\forall x \in \partial\Omega$.

The diffusive boundary conditions are more difficult to handle. Obviously, $f \equiv 0$ is a trivial solution of the problem. The homogeneity of the boundary condition suggests that a free parameter enters the problem, e.g., the mass $M = \int_\Omega \int_{\mathbb{R}^3} f(x, v)\, dv\, dx$. More about the choice of a free parameter can be found in Section 3.

**Remark 2.8.** For the free transport equation, i.e., Eq. (2.2) without collision integral, the diffusive boundary conditions were already studied in ref. 4.

In the following we prove partial results based on the theorems presented in Section 2.2. The idea is to investigate the so-called Albedo operator $\mathscr{A}$ which connects the in-and outgoing fluxes at the boundary $\partial\Omega$.

With the result of Section 2.2 we are able to obtain a solution $f(x, v)$ of the steady Boltzmann equation (2.2) by prescribing the ingoing flux $\Phi(x, v)$. Hence the Albedo operator $\mathscr{A}$ is well defined: suppose that $\Phi(x, v)$ is given for $x \in \partial\Omega$ and $v \in \mathbb{R}^3$ such that $v \cdot n(x) > 0$. Then we have a solution $f(x, v)$ on $\bar{\Omega} \times \mathbb{R}^3$ with

$$f(x, v) = \Phi(x, v), \qquad x \in \partial\Omega, \quad v \cdot n(x) > 0$$

The outgoing flux $f(x, v)$ at the boundary $\partial\Omega$ exactly defines the Albedo operator $\mathscr{A}$,

$$\mathscr{A}[\Phi](x, v) = f(x, v), \qquad x \in \partial\Omega, \quad v \cdot n(x) < 0$$

Now we are able to transform problem (2.2) with boundary condition (2.16) into a fixed-point problem: given the (inverse) temperature profile $\beta(x)$ on the boundary $\partial\Omega$, we are looking for a fixed point of the operator $\mathscr{R}$ defined by

$$(\mathscr{R}\Phi)(x, v) = m_{\beta(x)}(v) \int_{v_* \cdot n(x) < 0} |v_* \cdot n| \, \mathscr{A}[\Phi](x, v_*) \, dv_*$$

for $x \in \partial\Omega$ and $v \cdot n(x) > 0$.

**Remark 2.9.** Due to the bounds on the solution $f(x, v)$, the operator $\mathscr{R}$ is well defined.

If a fixed point $\Phi$ of $\mathscr{R}$ exists, then using the results of Section 2.2, we get a solution of (2.2) which fulfills the diffusive boundary conditions formulated in (2.16).

For simplicity we first consider the simpler problem of a free transport equation

$$v \cdot \nabla_x f = 0 \tag{2.18}$$

with boundary condition $f(x, v) = \Phi(x, v)$, $x \in \partial\Omega$, $v \cdot n(x) > 0$.

The solution is given by

$$f(x, v) = \Phi(z(x, v), v) \tag{2.19}$$

where $z(x, v)$ is the corresponding boundary point for $x \in \Omega$ following the characteristic line $\{x - sv, s > 0\}$.

**Remark 2.10.** Obviously (2.19) holds for arbitrary bounded domains $\Omega$ without any restriction on the size of $\Omega$, as discussed in the previous section.

Moreover, $\mathscr{A}$ is a linear operator with

$$\mathscr{A}[\Phi](x, v) = \Phi(z(x, v), v)$$

for $x \in \Omega$ and $v \cdot n(x) < 0$.

Now consider Eq. (2.18) together with the diffusive boundary condition (2.16). We are then looking for a fixed point of the operator $\mathscr{R}$ defined by

$$(\mathscr{R}\Phi)(x, v) = m_{\beta(x)}(v) \int_{v_* \cdot n < 0} |v_* \cdot n| \, \Phi(z(x, v_*), v_*) \, dv_* \qquad (2.20)$$

Because $\mathscr{A}$ is linear, one fixed point $\Phi$ of (2.20) directly leads to infinitely many fixed points by multiplying $\Phi$ with a nonnegative constant $c$. The operator $\mathscr{R}$ can be used to compute explicit solutions of the free-flow boundary value problem (2.18), as follows.

**Theorem 2.11.** Let $\Phi$ be a global Maxwellian

$$\Phi(x, v) = ce^{-\alpha v^2}, \qquad x \in \Omega, \quad v \in \mathbb{R}^3$$

and $\beta(x)$, $x \in \partial\Omega$, a given (inverse) temperature profile along the boundary $\partial\Omega$. Then $\mathscr{R}^2\Phi = \mathscr{R}\Phi$ and $\mathscr{R}\Phi$ is a fixed point of $\mathscr{R}$.

*Proof.* If $\Phi$ is a global Maxwellian, then

$$(\mathscr{R}\Phi)(x, v) = \frac{c\pi}{2\alpha^2} m_{\beta(x)} = \frac{c}{\alpha^2} \beta^2(x) \, e^{-\beta(x) v^2} \qquad (2.21)$$

and

$$\mathscr{R}(\mathscr{R}\Phi)(x, v) = m_{\beta(x)} \int_{v_* \cdot n < 0} |v_* \cdot n| \, \mathscr{A}[\mathscr{R}\Phi](x, v_*) \, dv_*$$

Using (2.21) yields

$$\mathscr{R}(\mathscr{R}\Phi)(x, v) = \frac{c}{\alpha^2} m_{\beta(x)} \int_{v_* \cdot n < 0} |v_* \cdot n| \, \beta^2(z(x, v_*)) \, e^{-\beta(z(x, v_*)) v_*^2} \, dv_* \qquad (2.22)$$

The right-hand side of (2.22) can be written as[4]

$$\int_{v_* \cdot n < 0} |v_* \cdot n| \, \beta^2 e^{-\beta v_*^2} \, dv_*$$

$$= \int_{e \cdot n, \, |e| = 1} |e \cdot n| \int_0^\infty |v_*|^3 \, \beta^2(y) \, e^{-\beta(y) |v_*|^2} \, d|v_*| \, de$$

where $y$ is the corresponding boundary point of $x \in \partial\Omega$ in direction $e$.

Substituting $\tau = \sqrt{\beta}\,|v_*|$ and calculating the remaining integrals gives

$$\int_{v_* \cdot n < 0} |v_* \cdot n|\, \beta^2 e^{-\beta v_*^2}\, dv_* = \frac{\pi}{2}$$

Hence,

$$\mathscr{R}(\mathscr{R}\Phi)(x, v) = \frac{c}{\alpha^2}\, \beta^2(x)\, e^{-\beta(x) v^2} = (\mathscr{R}\Phi)(x, v)$$

and $\mathscr{R}\Phi$ is a fixed point of $\mathscr{R}$, which completes the proof. ∎

The Albedo operator $\mathscr{A}$ of the Boltzmann equation is obviously nonlinear. However, if $\Phi$ is a fixed point of $\mathscr{R}$, then, multiplying $\Phi$ with an arbitrary constant $c \in \mathbb{R}_+$, we find that $c\Phi$ is a fixed point of the equation

$$v \cdot \nabla_x f = \frac{1}{c}\, Q(f, f)$$

together with boundary condition (2.16).

Because of the nonlinearity of $\mathscr{A}$, the existence of a fixed point of (2.20) is nontrivial. For a given flux $\Phi$, let

$$j(x) = \int_{v \cdot n < 0} |v \cdot n|\, \mathscr{A}[\Phi](x, v)\, dv$$

along the boundary $\partial\Omega$. Assuming that $\Phi$ is a fixed point of (2.20), we have that $\Phi$ must be of the form

$$\Phi(x, v) = j(x)\, m_{\beta(x)}(v), \qquad x \in \partial\Omega, \quad v \cdot n(x) > 0$$

Hence, we are looking for a function $j(x)$ such that

$$j(x) = (\mathscr{R}j)(x) = \int_{v \cdot n < 0} |v \cdot n|\, \mathscr{A}[j m_\beta](x, v)\, dv \tag{2.23}$$

**Remark 2.12.** For the free transport equation the fixed points of (2.23) are given by the constant functions, i.e., $j(x) = c \in \mathbb{R}_+$.

In the general multidimensional case, we cannot prove the existence of a $j$ satisfying (2.23). However, if we assume that a solution exists, we can prove some *a priori* estimates, as follows.

**Lemma 2.13.** Let $j \in L^1(\partial\Omega)$; then

$$\int_{\partial\Omega} (\mathscr{R}j)(x)\, d\sigma(x) = \int_{\partial\Omega} j(x)\, d\sigma(x)$$

*Proof.* Consider the problem

$$v \cdot \nabla_x f = Q(f, f) \tag{2.24}$$

with boundary condition $f(x, v) = j(x)\, m_{\beta(x)}(v)$ for all $x \in \partial\Omega$, $v \cdot n(x) > 0$. Integrating (2.24) with respect to $x$ and $v$ and applying the divergence theorem yields

$$\int_{\partial\Omega} \int_{v \cdot n > 0} |v \cdot n|\, j(x)\, m_{\beta(x)}(v)\, dv\, d\sigma(x)$$
$$= \int_{\partial\Omega} \int_{v \cdot n < 0} |v \cdot n|\, f(x, v)\, dv\, d\sigma(x) \tag{2.25}$$

Because of mass conservation at $\partial\Omega$, we have

$$\int_{\partial\Omega} \int_{v \cdot n > 0} |v \cdot n|\, j(x)\, m_{\beta(x)}(v)\, dv\, d\sigma(x) = \int_{\partial\Omega} j(x)\, d\sigma(x)$$

Hence, Eq. (2.25) reads

$$\int_{\partial\Omega} j(x)\, d\sigma(x) = \int_{\partial\Omega} \int_{v \cdot n < 0} |v \cdot n|\, \mathscr{A}[jm_\beta](x, v)\, dv\, d\sigma(x)$$
$$= \int_{\partial\Omega} (\mathscr{R}j)(x)\, d\sigma(x) \quad \blacksquare$$

The energy flux of $\mathscr{A}[jm_\beta](x, v)$ can be estimated as follows.

**Lemma 2.14.** Let $j \in L^1(\partial\Omega)$ and $\beta(x)$ be a founded (inverse) temperature profile along $\partial\Omega$. Then

$$\int_{\partial\Omega} \int_{v \cdot n < 0} |v \cdot n|\, v^2 \mathscr{A}[jm_\beta](x, v)\, dv\, d\sigma(x) = 2 \int_{\partial\Omega} j(x)/\beta(x)\, d\sigma(x)$$

*Proof.* Multiplying (2.24) by $v^2$ and integrating with respect to $x$ and $v$ yields

$$\int_\Omega \int_{\mathbb{R}^3} v v^2 \nabla_x f\, dx\, dv = 0 \tag{2.26}$$

Applying the divergence theorem, we find that Eq. (2.26) reads

$$\int_{\partial\Omega}\int_{v\cdot n>0} |v\cdot n|\, v^2 j(x)\, m_{\beta(x)}(v)\, dv\, d\sigma(x)$$

$$-\int_{\partial\Omega}\int_{v\cdot n<0} |v\cdot n|\, v^2 \mathscr{A}[jm_\beta](x,v)\, dv\, d\sigma(x)=0$$

Finally,

$$\int_{\partial\Omega}\int_{v\cdot n>0} |v\cdot n|\, v^2 j(x)\, m_{\beta(x)}(v)\, dv\, d\sigma(x)=2\int_{\partial\Omega} j(x)/\beta(x)\, d\sigma(x) \quad \blacksquare$$

In the one-dimensional case, $\Omega$ is an interval $[0, a], a>0$ on $\mathbb{R}$, we have exactly two boundary points and the lemma above reads

$$(\mathscr{R}j)(0)+(\mathscr{R}j)(a)=j(0)+j(a) \tag{2.27}$$

Because all terms in (2.27) are nonnegative, $\mathscr{R}$ is a mapping from $L_c=\{(x,y)\in\mathbb{R}^2, x+y=j\}$ into itself, where $j$ is an arbitrary positive constant describing the amount of mass flux into the interval $[0, a]$.

If $a$ is small enough (with respect to $j$) to apply the existence theorem of Section 2.2, we automatically get the existence of a fixed point, because $\mathscr{R}\colon L_c\to L_c$ is a continuous mapping.

**Theorem 2.15.** On a sufficiently small interval $[0, a]\subset\mathbb{R}$ there exists a solution of the boundary value problem (2.2) with diffusive boundary conditions (2.16).

The generalization to multidimensions remains an open question, due to the lack of local estimates on $(\mathscr{R}j)$.

## 3. GLOBAL EXISTENCE RESULTS IN THE ONE-DIMENSIONAL CASE

Global existence results for the steady one-dimensional slab problem were given in ref. 3 for discrete-velocity models and in ref. 2 for the full Boltzmann equation. The boundary conditions were either prescribed fluxes at both sides—used in refs. 2 and 3—or prescribed flux at one side and diffusive conditions at the other side.[3] In the following two sections it is shown how to generalize these results to the case of purely diffusive boundary conditions.

### 3.1. Discrete-Velocity Models

We recall the global existence result for discrete-velocity models as formulated in ref. 3. The discrete-velocity model in one space dimension is given by the set of ordinary differential equations

$$\xi_i \frac{df^i}{dx} = Q^i(f), \qquad i = 1,..., n \tag{3.1}$$

where $f = (f^1,..., f^n)$ are the particle densities associated with the $n$ admissible velocities $u_i \in \mathbb{R}^3$ and $\xi_i$ are the $x$-components of the vectors $u_i$. Each $Q^i$ has the form

$$Q^i(f) = \sum_{j,k,l} A^{ij}_{kl}(f^k f^l - f^i f^j)$$

such that conservation of mass, momentum, and energy is fulfilled.

**Remark 3.1.** The result given in ref. 3 even holds for more general types of collision terms $Q^i$. For example, conservation of energy is not required.

We consider (3.1) in the slab $0 < x < a$ under the additional assumption that $\xi_i \neq 0$ for all $i = 1,..., n$. In ref. 3 it was shown that (3.1) together with the boundary conditions

$$f^i(0) = \alpha^i, \qquad \xi_i > 0 \quad (\alpha^i \geqslant 0) \tag{3.2}$$

$$f^i(a) = \beta^i \sum{}^+ \xi_j f^j(a), \qquad \xi_i < 0 \quad (\beta^i \geqslant 0) \tag{3.3}$$

has a global solution for arbitrary slab length $a$. Here, $\sum^+$ ($\sum^-$) means that the sum is taken over all positive (negative) velocities $\xi_j$ and the coefficients $\beta_i$ are normalized such that $\sum^- \xi_i \beta^i = -1$.

**Theorem 3.2.**[3] The problem (3.1) with boundary conditions (3.2), (3.3) has a solution in $[\mathscr{C}^0_+]^n$.

Here $\mathscr{C}^0_+$ denotes the nonnegative, continuous functions in the interval $[0, a]$ and $[\mathscr{C}^0_+]^n$ is the Cartesian product of $n$ copies of $\mathscr{C}^0_+$.

The boundary conditions formulated in (3.2) and (3.3) are not exactly diffusive boundary conditions as given in Section 2.3 because the ingoing flux on the left-hand side of the slab is prescribed. Complete diffusive boundary conditions will be of the form

$$f^i(0)=\beta_0^i \sum^- \xi_j f^j(0), \qquad \xi_i>0 \tag{3.4}$$

$$f^i(a)=\beta_a^i \sum^+ \xi_j f^j(a), \qquad \xi_i<0 \tag{3.5}$$

with $\sum^+ \xi_i \beta_0^i = 1$ and $\sum^- \xi_i \beta_a^i = -1$.

However, the result formulated in ref. 3 can be used to show the existence of solutions of problem (3.1) together with the purely diffusive boundary conditions (3.4) and (3.5). As the boundary conditions (3.4), (3.5) are homogeneous in $f$, we expect a family of solutions (note that $f \equiv 0$ is a solution). It turns out that the outgoing flux $j^-(0)=\sum^- |\xi_j|\, f^i(0)$ can be chosen as a free parameter.

Denote by $j(x)$ the mass flux inside the slab, i.e., $j(x)=\sum \xi_i f^i(x)$, $j^+(x)=\sum^+ \xi_i f^i(x)$, and $j^-(x)=\sum^- |\xi_i| f^i(x)$. Because of mass conservation, every solution of (3.1) satisfies

$$\frac{dj(x)}{dx}=0$$

and with the boundary condition (3.3) we have $j(x)=0$ for all $0 \leqslant x \leqslant a$.

Now consider Eq. (3.1) with boundary conditions

$$f^i(0)=\beta_0^i \bar{j}^-(0), \qquad \xi_i>0 \tag{3.6}$$

$$f^i(a)=\beta_a^i \sum^+ \xi_j f^j(a), \qquad \xi_i<0 \tag{3.7}$$

where $\bar{j}^-(0)$ is some given positive constant, $\sum^+ \xi_i \beta_0^i = 1$, and $\sum^- \xi_i \beta_a^i = -1$. Due to the result of ref. 3 there exists a solution and

$$\frac{dj(x)}{dx}=0, \qquad j(a)=0$$

Hence $j(0)=0$ and $j^-(0)=\bar{j}^-(0)$, so we can state the following result.

**Theorem 3.3.** The problem (3.1) with boundary conditions (3.4), (3.5) has a one-parameter family of solutions in $[\mathscr{C}_+^0]^n$. The outgoing flux $j^-(0)$ parametrizes these solutions.

*Proof.* Consider Eq. (3.1) with boundary conditions (3.6), (3.7). By Theorem 3.2 the problem has a solution in $[\mathscr{C}_+^0]^n$. Because of $j^-(0)=\bar{j}^-(0)$, the solution also fulfills the boundary conditions (3.4), (3.5).

## 3.2. Measure Solutions in a Slab

In this section we consider the steady Boltzmann equation in a slab $0 \leqslant x \leqslant a$

$$\xi \frac{d}{dx} f = Q(f) \tag{3.8}$$

where $\xi$ denotes the $x$ component of the velocity $v$. our final goal will be to show the existence of a solution of (3.8) together with the diffusive boundary conditions

$$f(0, v) = j^{-}(0)\, m_0(v), \qquad \xi > 0 \tag{3.9}$$

$$f(a, v) = j^{+}(a)\, m_a(v), \qquad \xi < 0 \tag{3.10}$$

and $j^{-}(0) = \int_{\xi < 0} |\xi|\, f(0, v)\, dv, j^{+}(a) = \int_{\xi > 0} |\xi|\, f(a, v)\, dv$. Here, $m_0(v)$ and $m_a(v)$ are two (normalized) half-space Maxwellians,

$$m_0(v) = \frac{2\beta_0^2}{\pi} e^{-\beta_0 v^2} \qquad \text{and} \qquad m_a(v) = \frac{2\beta_a^2}{\pi} e^{-\beta_a v^2} \tag{3.11}$$

In ref. 2 an existence result for problem (3.8) together with the boundary conditions

$$f(0, v) = f_0(v) \tag{3.12}$$

$$f(a, v) = f_a(v) \tag{3.13}$$

was given. The solution was found in the space of measure-valued functions of $x$,

$$x \to \mu_x, \qquad [0, a] \to M$$

where $M$ denotes the set of bounded measures on $\mathbb{R}^3$. We outline the strategy followed in ref. 2, without giving all the technical details.

The singlė steps to the existence result are the following: first of all one passes to the measure formulation of problem (3.10), which yields the equation

$$\xi \frac{d}{dx} \mu_x = Q(\mu_x, \mu_x) \tag{3.14}$$

in the sense of weak*-convergence of measures. The collision kernel is assumed to be of the form (1.2) with $-1 \leqslant k \leqslant 0$. The kernel is first truncated in the same way as in Section 2.1 and further by a crude truncation of the form

$$B^{\delta} = B_{\varepsilon} k_{\delta}$$

with

$$k_{\delta} = \begin{cases} 1 & \text{if } v^2 + v_*^2 \leqslant \delta^{-2},\ \min\{|\xi|, |\xi_*|, |\xi'|, |\xi'_*|\} > \delta, \text{ or } |v - v_*| > \delta \\ 0 & \text{otherwise} \end{cases}$$

so that Eq. (3.14) is replaced by

$$\xi \frac{d}{dx} \mu_x = Q^{\delta}(\mu_x, \mu_x) \tag{3.15}$$

with boundary conditions $\mu_0|_{\{\xi > 0\}} = \mu_0^+$ and $\mu_a|_{\{\xi < 0\}} = \mu_a^-$. In the following we will use the notation $\|\mu_.\| = \sup_{x \in [0, a]} \int d\mu_x(v)$.

The collision operator $Q^{\delta}(\mu_x, \mu_x)$ can be written in the form

$$Q^{\delta}(\mu_x, \mu_x) = Q_+^{\delta}(\mu_x, \mu_x) - L^{\delta}(\mu_x)\, \mu_x$$

where $Q_+^{\delta}(\mu_x, \mu_x)$, $L^{\delta}(\mu_x)\, \mu_x$ are measures defined by

$$\langle Q_+^{\delta}(\mu_x, \mu_x), \varphi \rangle = \int_v \int_{v_*} \int_n B^{\delta}(v, n, v_*)\, \varphi(v)\, d(M_x \circ J)$$

with $dM_x = d\omega(n)\, d\mu_x \times d\mu_x$ and

$$\langle L^{\delta}(\mu_x)\, \mu_x, \varphi \rangle = \langle \mu_x, L^{\delta}(\mu_x)\, \varphi \rangle$$

Because of the truncation of $B$, we have

$$\|L^{\delta}(\mu_x)(x)\| \leqslant 4\pi C(\delta) \int d\mu_x(w)$$

with $B^{\delta} \leqslant C(\delta)$.

Let $X = (\mathscr{C}[0, a]; M)$ be the cone of all continuous functions $[0, a] \to M$ and $B_R(0) \subset X$ the set of all continuous measure-valued functions $\mu_.$ such that $\|\mu_.\| \leqslant R$. For $\tau \geqslant 4\pi C(\delta)$ one studies the operators

$$T(\tau)\colon\quad B_R(0) \to X, \quad v_. = T(\tau)\, \mu \tag{3.16}$$

defined by

$$\nu_0|_{\{\xi>0\}}=\mu_0^+, \qquad \nu_a|_{\{\xi<0\}}=\mu_a^- \tag{3.17}$$

$$\xi \frac{d}{dx}\nu_x=0 \tag{3.18}$$

for $|\xi|\leqslant\delta$ and $|\xi|\geqslant 1/\delta$ and

$$\xi \frac{d}{dx}\nu_x+\tau\rho[\mu_.](x)\,\nu_x=Q_+^\delta(\mu_x,\mu_x)+\tau\rho[\mu_.](x)\,\mu_x-L^\delta(\mu_x)\,\mu_x \tag{3.19}$$

for $\delta<|\xi|<1/\delta$, $\rho[\mu_.]=\int d\mu_.(v)$.

The boundary value problem (3.17)–(3.19) has a unique solution $\nu_.$, and the mapping $\mu_.\to\nu_.$ is continuous from $B_R(0)$ into $X$. Because $T(\tau)$ will in general not map $B_R(0)$ into itself, one introduces the retract $T_R$: $X\to B_R(0)$, defined by

$$(T_R\mu_.)_.=\begin{cases}\mu_. & \text{if } \|\mu_.\|\leqslant R\\ \dfrac{R}{\|\mu_.\|}\mu_. & \text{if } \|\mu_.\|>R\end{cases} \tag{3.20}$$

Now $T_R\circ T(\tau)\,B_R(0)$ is relatively compact and has a fixed point in $B_R(0)$. To prove the existence of a fixed point of $T(\tau)$ in some $B_R(0)$ it remains to show that the set of all fixed points of $T_R\circ T(\tau)$ is uniformly bounded. This was achieved using the conservation quantities of the Boltzmann equation. Finally it was shown—using the usual Cantor diagonalization—how to get rid of the crude truncation.

In order to handle diffusive boundary conditions (3.9), (3.10) we first generalize the existence result given above to the case where the boundary condition (3.13) is replaced by general stochastic scattering conditions of the form

$$|\xi|\,f(a,v)=\int_{\xi'>0}R_a(v'\to v)\,|\xi'|\,f(a,v)\,dv$$

with an appropriate boundary kernel $R_a$. In a second step we show —similar to the consideration in the previous section—how to extract the case of purely diffusive boundary conditions (3.9), (3.10).

As in ref. 2, we consider the measure formulation of problem (3.8) together with the boundary conditions

$$\mu_0|_{\{\xi>0\}}=\mu_0^+ \tag{3.21}$$

$$\mu_a|_{\{\xi<0\}}=\int_{\xi'>0}R_a(v'\to v)\frac{|\xi'|}{|\xi|}\,d\mu_a(v) \tag{3.22}$$

Here, the boundary kernel $R_a(v' \to v)$ should fulfill the following conditions:[5]

$$R \geqslant 0 \tag{3.23}$$

$$\int_{\xi<0} R_a(v' \to v)\, dv = 1 \tag{3.24}$$

(mass conservation),

$$\exists c_1 > 0 \qquad \text{such that} \quad \int_{\xi<0} R_a(v' \to v)\, |\xi|\, dv \geqslant c_1 \tag{3.25}$$

("spreading condition"), and

$$\exists c_2 > 0 \qquad \text{such that} \quad \int_{\xi<0} R_a(v' \to v)\, v^2\, dv \leqslant c_2 \tag{3.26}$$

("energy condition").

Using the same truncation $B^\delta$ of the collision kernel as above, we define the operator $T(\tau)$ as in (3.16), where the boundary conditions (3.17) are replaced by (3.21). Following the same analysis as in ref. 2, we are able to prove the existence of a fixed point of $T_R \circ T(\tau)\, B_R(0)$, where $T_R$ is the retract defined in (3.20). As mentioned above, it remains to prove that the set of all fixed points of $T_R \circ T(\tau)$ is uniformly bounded. This can be done following similar arguments as in ref. 2:

**Lemma 3.4.** For any solution $\mu^\tau_\cdot$ of the equation

$$T_R \circ T(\tau)\, \mu^\tau_\cdot = \mu^\tau_\cdot$$

with boundary conditions (3.21) we have

$$\max_{x \in [0,a]} \int \xi^2\, d\mu^\tau_x(v) \leqslant C(\mu_0^+)$$

*Proof.* Let $j^+(x) = \int_{\xi>0} \xi\, d\mu_x(v)$, $j^-(x) = \int_{\xi<0} |\xi|\, d\mu_x(v)$, $j = j^+ - j^-$, and $p^+(x) = \int_{\xi>0} \xi^2\, d\mu_x(v)$, $p^-(x) = \int_{\xi<0} \xi^2\, d\mu_x(v)$, $p = p^+ + p^-$; then by the usual conservation laws,

$$\int \varphi(v)\, d[Q^\delta_+(\mu_x, \mu_x) - L^\delta(\mu_x)\, \mu_x](v) = 0$$

where $\varphi(v) = 1$, $v$, or $v^2$. Hence, if $\mu^\tau_\cdot$ is a fixed point of $T_R \circ T(\tau)$,

$$\frac{dj}{dx} = (\lambda - 1)\,\tau p^2$$

and

$$\frac{dp}{dx} = (\lambda - 1)\,\tau p j \tag{3.27}$$

where $\lambda = \min\{R/\|T(\tau)\,\mu^\tau_\cdot\|, 1\}$.

Because $\rho(x) \geqslant 0$, we have $j(x) = j^+(x) - j^-(x) \leqslant j^+(0) - j^-(0)$ and

$$\begin{aligned} j(a) &= j^+(a) - j^-(a) \\ &= \int_{\xi > 0} \xi\, d\mu_x(v) - \int_{\xi < 0} |\xi|\, d\mu_x(v) \\ &= \int_{\xi > 0} \xi\, d\mu_x(v) - \int_{\xi < 0} \int_{\xi' > 0} R_a(v' \to v)\, |\xi'|\, d\mu_x(v')\, dv \\ &= 0 \end{aligned}$$

Therefore, $j(x) \geqslant 0$ for all $0 \leqslant x \leqslant a$ and especially $j^-(0) \leqslant j^+(0)$. Furthermore, by (3.27), $p$ is nonincreasing and we just need to find an estimate on $p(0) = p^+(0) + p^-(0)$.

The ingoing momentum flux $p^+(0)$ is given by the boundary condition at $x = 0$, so it remains to estimate $p^-(0)$. By the Cauchy–Schwarz inequality,

$$\begin{aligned} p^-(0) &= \int_{\xi < 0} |\xi|^2\, d\mu_0(v) \\ &\leqslant \left[\int_{\xi < 0} |\xi|\, d\mu_0(v)\right]^{1/2} \left[\int_{\xi < 0} |\xi|^3\, d\mu_0(v)\right]^{1/2} \\ &\leqslant [j^+(0)]^{1/2}\,[q^-(0)]^{1/2} \end{aligned} \tag{3.28}$$

where $q^+(x) = \int_{\xi > 0} \xi v^2\, d\mu_x(v)$, $q^-(x) = \int_{\xi < 0} |\xi|\, v^2\, d\mu_x(v)$, and $q(x) = q^+(x) - q^-(x)$. Furthermore, the energy flux $q$ fulfills

$$\frac{dq}{dx} = (\lambda - 1)\,\tau p e \leqslant 0, \qquad e = \int v^2\, d\mu_x(v)$$

and

$$q^+(a) + q^-(0) \leqslant q^+(0) + q^-(a)$$

Hence, from (3.28)

$$p^-(0) \leqslant [j^+(0)]^{1/2} [q^+(0) + q^-(a)]^{1/2}$$

Now

$$q^-(a) \leqslant c_2 j^+(a) \tag{3.29}$$

because of (3.26) and

$$j^+(a) \leqslant \frac{1}{c_1} p^+(a) \tag{3.30}$$

because of the "spreading condition" (3.25). Using (3.29), (3.30), and $p(a) \leqslant p(0)$ yields

$$p^-(0) \leqslant [j^+(0)]^{1/2} \left\{ q^+(0) + \frac{c_2}{c_1} [p^+(0) + p^-(0)] \right\}^{1/2}$$

Recognizing that

$$ab \leqslant \frac{1}{\varepsilon} a^2 + \varepsilon b^2, \qquad \forall \varepsilon, a, b > 0$$

we get

$$p^-(0) \leqslant \frac{1}{\varepsilon} j^+(0) + \varepsilon \left[ q^+(0) + \frac{c_2}{c_1} p^+(0) \right] + \varepsilon \frac{c_2}{c_1} p^-(0), \qquad \forall \varepsilon > 0$$

from which we conclude that $p^-(0)$ is bounded. ∎

We consider the problem (3.15) together with the (partial) diffusive boundary conditions

$$\mu_0 |_{\{\xi > 0\}} = \int_{\xi' < 0} m_{\beta_0}(v) \, |\xi'| \, d\mu_0(v') \tag{3.31}$$

$$\mu_a |_{\{\xi < 0\}} = \int_{\xi' > 0} R_a(v' \to v) \frac{|\xi'|}{|\xi|} \, d\mu_a(v') \tag{3.32}$$

Here, $m_0(v)$ is a normalized Maxwellian with (inverse) temperature $\beta_0$ as given in (3.11) and $R_a$ should satisfy the conditions (3.23)–(3.26). This is indeed the case for the conditions (3.9) and (3.10), as pointed out in ref. 6.

**Theorem 3.5.** For any $\delta > 0$, the problem (3.15) together with the boundary conditions (3.31), (3.32) has a one-parameter family of measure solutions $\mu_{\dots} \in X$. These solutions can be parametrized by the outgoing mass flux at $x = 0$.

*Proof.* Let $\mu^{\tau}_{\cdot}$ be as in Lemma 3.4; then

$$\|\mu^{\tau}_{\cdot}\| \leqslant \sup_{x \in [0, a]} \left( \int_{\substack{0 < \xi < \delta \\ 1/\delta < \xi}} d\mu_0^+ + \int_{\substack{-\delta < \xi < 0 \\ -1/\delta > \xi}} \int_{\xi' > 0} R_a(v' \to v) \frac{|\xi'|}{|\xi|} d\mu_0^+ \, dv + \frac{1}{\delta^2} \int \xi^2 \, d\mu^{\tau}_x \right)$$

$$\leqslant C_1(\delta)$$

Here, $C_1(\delta)$ is a constant which depends on the boundary values and on $\delta$, but not on $\tau$ and $R$. Then, by the same argument as in ref. 2, one concludes that for $\tau_1 = 4\pi C(\delta)$ there exists a fixed point $\mu^{\tau_1}_{\cdot}$ of $T(\tau_1)$, i.e., $T(\tau_1)\, \mu^{\tau_1}_{\cdot} = \mu^{\tau_1}_{\cdot}$.

Now we consider at $x = 0$ the (prescribed) measure $\mu_0^+$ with density $\bar{j}^-(0)\, m_0(v)$, where $\bar{j}^-(0)$ is some positive constant and $m_0$ is defined as in (3.31). What remains to show is that the corresponding fixed point $\mu^{\tau_1}_{\cdot}$ fulfills the condition

$$\int_{\xi < 0} |\xi| \, d\mu_0 = \bar{j}^-(0)$$

This can be done with the same arguments as in the previous subsection. Because $\mu^{\tau_1}_{\cdot}$ is a fixed point of $T(\tau_1)$, we get by mass conservation the equation

$$\frac{dj}{dx} = 0$$

where $j(x) = \int \xi \, d\mu^{\tau_1}_x$. At $x = a$ we have, because of the mass conservation of the boundary kernel, $j(a) = 0$, from which we conclude that $j(x) = 0$ for all $x \in [0, a]$. Hence, at $x = 0$, this yields

$$j^-(0) = \int_{\xi < 0} |\xi| \, d\mu_0^{\tau_1} = \int_{\xi > 0} \xi \, d\mu_0^+ \tag{3.33}$$

Because $\mu_0^+$ is the measure with density $\bar{j}^-(0)\, m_0(v)$, we get

$$j^-(0) = \int_{\xi > 0} \bar{j}^-(0)\, \xi m_0(v) \, dv = \bar{j}^-(0)$$

Hence, the fixed point $\mu^{\tau_1}$ is a solution which fulfills the boundary conditions (3.31), (3.32).

Moreover, because $j^-(0)$ is some positive constant, there exists a one-parameter family of solutions for fixed boundary conditions [defined by the (inverse) temperature $\beta_0$ and the boundary kernel $R_a$] and $j^-(0)$ parametrizes this family. ∎

**Remark 3.6.** The result given in Theorem 3.5 is more general than the problem formulated at the beginning, i.e., the steady Boltzmann equation (3.8) together with the boundary conditions (3.9), (3.10). At $x = a$ an arbitrary boundary kernel $R_a$ which satisfies the conditions (3.23)–(3.26) can be used.

## 4. FURTHER RESULTS AND CONCLUDING REMARKS

In two or more dimensions and without smallness conditions on the domain and truncations of the collision kernel, nothing about existence of solutions is known.

We mention, without proof, two *a priori* estimates on solutions which can be proved along the lines given in ref. 7, by careful use of the invariants. Suppose $f$ solves

$$v \nabla_x f = Q(f) \tag{4.1}$$

on the bounded domain $\Omega \subset \mathbb{R}^n$, $n = 1$, 2, or 3. Then we can prove the following.

**Proposition 4.1.** Consider Eq. (4.1) together with a boundary condition $f(x, v) = \Phi(x, v)$ for $x \in \partial\Omega$, $v \cdot n(x) > 0$ and assume that the ingoing mass and energy flux are bounded. Then, for every solution $f(x, v)$ of (4.1),

$$\int_S \int_{v \cdot n < 0} |v \cdot n(x)| \, f(x, v) \, dv \, d\sigma(x) \leqslant c_1$$

$$\int_S \int_{v \cdot n < 0} |v \cdot n(x)| \, v^2 f(x, v) \, dv \, d\sigma(x) \leqslant c_2$$

for all $S \subset \partial\Omega$ measurable. Here, $c_1$ and $c_2$ are two constants which depend only on the boundary values.

**Proposition 4.2.** Consider the steady Boltzmann equation (4.1) with boundary condition $f(x, v) = \Phi(x, v)$ such that the ingoing mass and energy flux are bounded. Let $H_0'$ be a hyperplane with normal vector $n_0$ and $H_0 = H_0' \cap \Omega \neq \emptyset$. Then, for every solution $f(x, v)$ of (4.1),

$$\int_{H_0} \int_{\mathbb{R}^3} (v \cdot n_0)^2 f(x, v) \, dv \, d\sigma(x) \leqslant C$$

where $C$ is a constant which depends only on the boundary values.

Finally, we mention that numerous numerical examples of solutions of steady boundary value problems for the Boltzmann equation are given in ref. 9. For more background on the numerical tools and their application, we refer to ref. 8 and 10.

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## NOTE ADDED IN PROOF

We recently noticed that some results for steady boundary value problems for large Knudsen numbers, but *without* trancations of the collision kernel, are contained in: Nonlinear Evolution Equations, by Nina B. Maslova, World Scientific Singapore 1993. Truncations in the collision kernel are avoided there by the use of suitable weight functions in the norms.

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